Q. How does the arithmetic and calculus of power series (as functions) work?

Thm: (Combining/Modifying Power Series) Suppose $f(x) = \sum_{n=0}^{\infty} c_n x^n$ converges for all x in $(-R_1, R_1)$ and $g(x) = \sum_{n=0}^{\infty} d_n x^n$ converges for all x in $(-R_2, R_2)$. Let $R = \min(R_1, R_2)$. Then

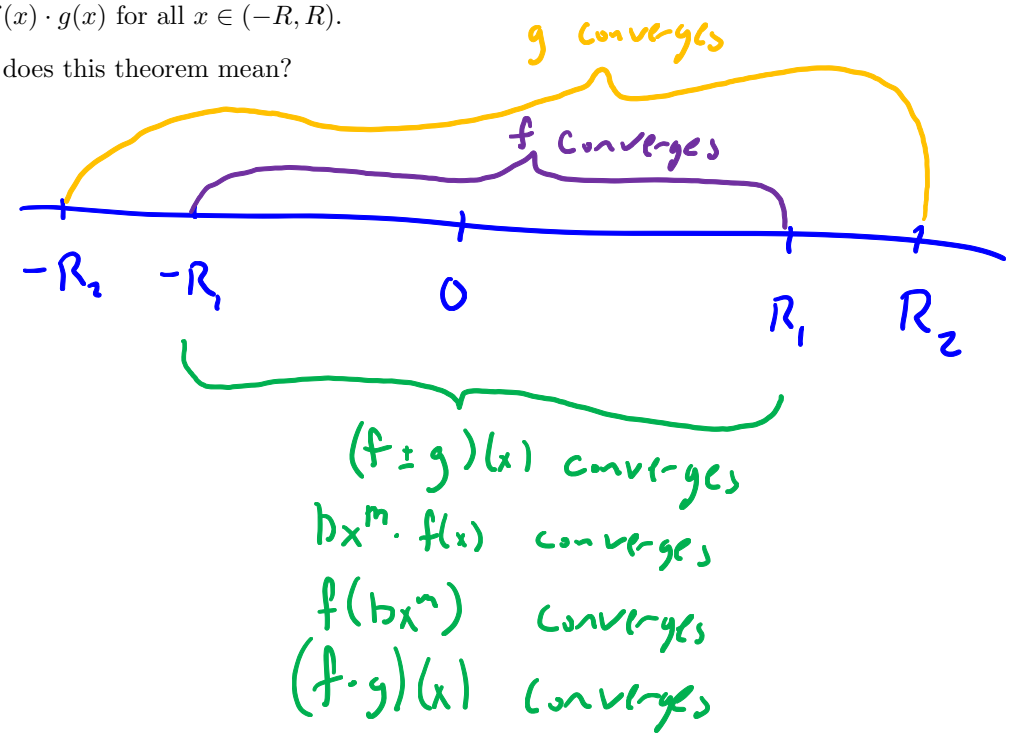
(a) The series $\sum_{n=0}^{\infty} (c_n \pm d_n)x^n$ converges to $f(x) \pm g(x)$ for all $x \in (-R, R)$.

(b) Given a constant b and an integer $m \geqslant 0$, the series $\sum_{n=0}^{\infty} bc_n x^{n+m} = \sum_{n=0}^{\infty} (bx^m)c_n x^n$ converges to $bx^m f(x)$ for all x in $(-R, R)$.

(c) Given a constant b and an integer $m \geqslant 0$, the series $\sum_{n=0}^{\infty} b^n c_n x^{mn} = \sum_{n=0}^{\infty} c_n (bx^m)^n$ converges to $f(bx^m)$ for all x such that $bx^m \in (-R, R)$.

(d) If $e_n = \sum_{k=0}^{n} c_k d_{n-k} = c_0 d_n + c_1 d_{n-1} + \cdots + c_n d_0$, then the series $\sum_{n=0}^{\infty} e_n x^n$ converges to the product $f(x) \cdot g(x)$ for all $x \in (-R, R)$.

Q. What does this theorem mean?



Moral: Arithmetic works the way you want it to when all involved series converge

Why is (d) true?

$$\left(\sum_{n=0}^{\infty} c_n x^n\right)\left(\sum_{n=0}^{\infty} d_n x^n\right)$$

$$= \left(c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots\right)\left(d_0 + d_1 x + d_2 x^2 + d_3 x^3 + \cdots\right)$$

$$= c_0 d_0 + c_0 d_1 x + c_0 d_2 x^2 + c_0 d_3 x^3 + \cdots$$
$$+ c_1 d_0 x + c_1 d_1 x^2 + c_1 d_2 x^3 + \cdots$$
$$+ c_2 d_0 x^2 + c_2 d_1 x^3 + \cdots$$
$$+ c_3 d_0 x^3 + \cdots$$

$$= c_0 d_0 + \left(c_0 d_1 + c_1 d_0\right) x + \left(c_0 d_2 + c_1 d_1 + c_2 d_0\right) x^2$$
$$+ \left(c_0 d_3 + c_1 d_2 + c_2 d_1 + c_3 d_0\right) x^3$$
$$+ \cdots$$

Ex 1 Suppose $f(x) = \dfrac{1}{5-x}$ and $g(x) = \dfrac{1}{x-2}$.

(a) Find the power series for f and g centered at 0.

$$f(x) = \frac{1}{5-x} = \frac{1}{5} \cdot \frac{1}{1-\frac{x}{5}}$$

$$= \frac{1}{5} \sum_{n=0}^{\infty} \left(\frac{x}{5}\right)^n$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{5^{n+1}}$$

$$g(x) = \frac{1}{x-2} = \frac{1}{-2} \frac{1}{-\frac{x}{2}+1} = \frac{1}{-2} \cdot \frac{1}{1-\frac{x}{2}}$$

$$= -\frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n$$

$$= \sum_{n=0}^{\infty} -\frac{x^n}{2^{n+1}}$$

(b) Use part (a) to find the power series for $\frac{1}{5-3x^7}$ centered at 0.

$$\frac{1}{5-3x^7} = f(3x^7) = \sum_{n=0}^{\infty} \frac{(3x^7)^n}{5^{n+1}}$$

$$= \sum_{n=0}^{\infty} \frac{3^n \cdot x^{7n}}{5^{n+1}}$$

(c) Use part (a) to find the power series for $\frac{3x}{x^2-2}$ centered at 0.

$$\frac{3x}{x^2-2} = 3x \cdot \frac{1}{x^2-2} = 3x \cdot g(x^2)$$

$$= 3x \cdot \sum_{n=0}^{\infty} -\frac{(x^2)^n}{2^{n+1}}$$

$$= 3x \cdot \sum_{n=0}^{\infty} -\frac{x^{2n}}{2^{n+1}}$$

$$= \sum_{n=0}^{\infty} -\frac{3x^{2n+1}}{2^{n+1}}$$

(d) Write out the first 3 terms of the series for $\frac{1}{(5-x)(x^2-2)}$ centered at 0.

$$\frac{1}{(5-x)(x^2-2)} = \frac{1}{5-x}\cdot\frac{1}{x^2-2}$$

$$= \left(\sum_{n=0}^{\infty}\frac{x^n}{5^{n+1}}\right)\left(\sum_{n=0}^{\infty}\frac{-3x^{2n+1}}{2^{n+1}}\right)$$

$$= \left(\frac{1}{5}+\frac{x}{25}+\frac{x^2}{125}+\frac{x^3}{675}+\cdots\right)\left(\frac{-3x}{2}-\frac{3x^3}{4}-\cdots\right)$$

$$= \left(\frac{1}{5}\cdot\frac{-3x}{2}-\frac{1}{5}\cdot\frac{3x^3}{4}-\cdots\right)$$

$$+\left(\frac{x}{25}\cdot\frac{-3x}{2}-\cdots\right)$$

$$+\left(\frac{x^2}{125}\cdot\frac{-3x}{2}-\cdots\right)$$

$$= -\frac{3}{10}x-\frac{3}{50}x^2-\left(\frac{3}{20}+\frac{3}{250}\right)x^3-\cdots$$

Ex 3 Find the series $\frac{1}{(5-x)(x-2)}$ centered at 0 using partial fraction decomposition.

$$\frac{1}{(5-x)(x-2)} = \frac{A}{5-x} + \frac{B}{x-2} \rightarrow 1 = A(x-2) + B(5-x)$$

$x=2 \rightarrow 1 = 3B \rightarrow B = \frac{1}{3}$

$x=5 \rightarrow 1 = 3A \rightarrow A = \frac{1}{3}$

$$\frac{1}{(5-x)(x-2)} = \frac{1/3}{5-x} + \frac{1/3}{x-2}$$

$$= \frac{1}{3}\sum_{n=0}^{\infty}\frac{x^n}{5^{n+1}} + \frac{1}{3}\cdot\sum_{n=0}^{\infty}\frac{-x^n}{2^{n+1}}$$

$$= \sum_{n=0}^{\infty}\frac{x^n}{3\cdot 5^{n+1}} - \sum_{n=0}^{\infty}\frac{x^n}{3\cdot 2^{n+1}}$$

$$= \sum_{n=0}^{\infty}\frac{x^n}{3\cdot 5^{n+1}} - \frac{x^n}{3\cdot 2^{n+1}}$$

$$= \sum_{n=0}^{\infty}\left(\frac{1}{3\cdot 5^{n+1}} - \frac{1}{3\cdot 2^{n+1}}\right)x^n$$

Q. Suppose that $\sum_{n=0}^{\infty} c_n x^n$ has radius of convergence R. What is the radius of convergence of $\sum_{n=0}^{\infty} n \cdot c_n x^{n-1}$? What about $\sum_{n=0}^{\infty} \frac{c_n}{n+1} x^{n+1}$? Can you tell anything about the intervals of convergence for these two series?

$\sum_{n=0}^{\infty} c_n x^n$ has R.o.C. R

↳ $\lim_{n\to\infty} \left| \frac{c_{n+1} x^{n+1}}{c_n x^n} \right| = \lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \cdot x \right|$

$= |x| \cdot \lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right| < 1$

$= \frac{1}{R}$ makes R.o.C. $= R$

$\sum_{n=0}^{\infty} n c_n x^{n-1}$

↳ $\lim_{n\to\infty} \left| \frac{(n+1) c_{n+1} x^n}{n \cdot c_n \cdot x^{n-1}} \right| = |x| \cdot \lim_{n\to\infty} \left| \frac{(n+1)}{n} \cdot \frac{c_{n+1}}{c_n} \right|$

$\frac{(n+1)}{n} \to 1$, $\frac{c_{n+1}}{c_n} \to \frac{1}{R}$

↳ R.o.C. $= R$

Same for $\sum_{n=0}^{\infty} \frac{c_n}{n+1} x^{n+1}$

Thm: (Term-by-term Differentiation and Integration) Suppose $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ converges for all $x \in (a-R, a+R)$. Then f is differentiable and integrable for all x in $(a-R, a+R)$, and

$\frac{d}{dx}\left(\sum_{n=0}^{\infty} c_n (x-a)^n \right) = \frac{d}{dx}\left(c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots \right)$

$= \left(c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots \right)$

$= \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$

for all $x \in (a-R, a+R)$.

$\int$ stuff $= C + \sum \cdots$

Ex 4 Find the series for $f(x) = \frac{1}{(1-x)^2}$ centered at 0.

Note: $\int \frac{1}{(1-x)^2}\,dx = \frac{1}{1-x} + C$

$\rightarrow \int \frac{1}{(1-x)^2}\,dx = C + 1 + x + x^2 + x^3 + \cdots$

$\rightarrow \frac{1}{(1-x)^2} = \frac{d}{dx}\left(C + 1 + x + x^2 + x^3 + \cdots\right)$

$= 1 + 2x + 3x^2 + \cdots$

$= \sum_{n=0}^{\infty} (n+1)\cdot x^n$

Ex 5 Find the series for $f(x) = \ln|1-x|$ centered at 0.

$\frac{df}{dx} = \frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$

$\rightarrow f(x) = \int \frac{1}{1-x}\,dx = C + x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots$

$0 = \ln|1-0| = f(0) = C + 0 + \frac{0^2}{2} + \frac{0^3}{3} + \cdots = C$

$\hookrightarrow f(x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots = \sum_{n=1}^{\infty} \frac{x^n}{n}$

Ex 6 Use power series to estimate ln(2).

Note: $x=1 \longleftrightarrow \sum_{n=1}^{\infty} \frac{1}{n} \longleftrightarrow$ "$\ln(0)$"

$x=-1 \longleftrightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \longleftrightarrow$ "$f(-1) = \ln(2)$"

Problem: theorem only works for $x \in (-1, 1)$

Trick: $\ln(2) = -\ln(2^{-1}) = -\ln(\frac{1}{2})$

$$= -\ln\left|1 - \frac{1}{2}\right| = -\sum_{n=1}^{\infty} \frac{(1/2)^n}{n} = -\sum_{n=1}^{\infty} \frac{1}{n \cdot 2^n}$$

$$= -\left(\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 2^2} + \frac{1}{3 \cdot 2^3} + \cdots\right)$$